

# Restricted Galileons

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## Abstract

We study Galileon theories that emerge in ghost-free massive gravity. In particular, we focus on a sub-class of these theories where the Galileons can be completely decoupled from the tensor Lagrangian. These Galileons differ from generic ones – they have interrelated coefficients of the cubic and quartic terms, and most importantly, a non-standard coupling to external stress-tensors, governed by the same coefficient. We show that this theory has no static stable spherically symmetric solutions that would interpolate from the Vainshtein region to flat space; these two regions cannot be smoothly matched for the sign of the coefficient for which fluctuations are stable. Instead, for this sign choice, a solution in the Vainshtein domain is matched onto a cosmological background. Small fluctuations above this solution are stable, and sub-luminal. We discuss observational constraints on this theory, within the quantum effective Lagrangian approach, and argue that having a graviton mass of the order of the present-day Hubble parameter, is consistent with the data. Last but not least, we also present a general class of cosmological solutions in this theory, some of which exhibit the de-mixing phenomenon, previously found for the self-accelerated solution.

# 1 Introduction and summary

In the present work we will be interested in those Galileons that describe helicity-0 modes of tensor fields: In the context of the DGP model [1] the cubic Galileon was found in Ref. [2], and was shown to describe properties of the helicity-0 mode. More general, quartic and quintic Galileons, were studied in [3].

Most relevant to the present work, is the fact that all the Galileons appear in the Lagrangian for the helicity-0 mode of a massive graviton [4, 5] (see [6] for earlier evidence). Although the massive gravity Galileons are similar to the generic ones [3], there is a nuance that distinguishes them: the massive gravity Galileons come with specific coefficients, and most importantly, with a novel coupling to the matter stress-tensor governed by the same coefficient [6, 4, 7].

One motivation for this work is to study spherically symmetric solutions in this special class of massive gravity Galileons. In particular, in the present work we will focus on the sub-class of these theories for which the helicity-0 and helicity-2 modes can be diagonalized [4]. We refer to this sub-class as Restricted Galileons.

Spherically symmetric solutions in general Galileon theories are well understood [2]: generically, one gets a solution that transitions from the Vainshtein region to the asymptotically flat one; furthermore, typically, stable fluctuations, both in and outside of the Vainshtein region [8], exhibit super-luminal propagation [3]. This however, does not necessarily lead to acausality – attempts to create closed time-like curves bring one beyond the domain of validity of an effective field theory [9]. Pending a classical, or quantum, completion of these theories at the strong scale, it makes sense to postpone the discussion of this issue. For a review of other interesting theoretical constructions with Galileons, see Refs. [10].

We will show that, there are significant differences in the case of restricted Galileons (ReG). In particular, we will prove that in ReG no stable static spherically symmetric solutions exist that would interpolate between the Vainshtein region to an asymptotically flat domain. Instead, the Vainshtein region is naturally matched onto a solution that does not asymptote to flat space, but does to a cosmological background. The perturbations on this solution are stable, and sub-luminal!

For the exact static solution, we address the issue of the bound on the graviton mass, as inferred from it. We show that having the graviton mass as small as the value of the Hubble parameter today is consistent with the bounds derived from the table-top sub-millimeter measurements of gravity-competing forces. Our conclusions differ from those in [11], and we discuss the origin of this difference.

The exact static solution, if applied to distribution of sources like the ones in our Universe (galaxies, clusters, etc), is consistent with the observations, due to the fact that the Vainshtein radius in such a universe gets pushed – statistically speaking – closer to the Hubble scale. There may however be isolated clusters in the universe for which beyond-the-Vainshtein-scale physics can be explored.

One could ask what would happen to a dilute distribution of matter that initially has no Vainshtein region, which is placed in an asymptotically flat space, and is

being adiabatically collapsed to form a smaller object with higher densities. Then, the energy density and pressure in the nonlinear helicity-0 field blow up when the size of the distribution becomes of the order of the Vainshtein radius, preventing such an adiabatic collapse. Thus, we'd expect the respective spherically symmetric collapsing solution, if it exists at all, to have time dependence that is fast as compared to the Vainshtein scale. In Appendix A we also present a class of more general time-dependent solutions, however, non of them reach an asymptotically flat space for any constant time slice.

Three important comments on the literature:

(A) All the results of the present work are obtained in the decoupling limit. As to whether our results apply to the solutions of the full theory, discussed in Refs. [12, 13, 14, 15, 16], remains to be seen.

(B) Refs. [17, 18] have recently shown the existence of superluminal shocks for some parameter space of massive gravity. Such superluminal solutions, like in Galileon theories, appear well-within the effective field theory domain. However, as briefly discussed above in the 4th paragraph of this section, and in detail in Ref. [9], the question whether the corresponding solutions can lead to acausal stable closed timelike curves, needs to be addressed including perturbations above these solutions, in the context of an effective field theory. In cases considered so far, the fluctuations above such solutions bring one beyond the effective field theory [9] – one more argument that these theories need either calculational prescription, or classical/quantum completion below/at the strong scale<sup>1</sup>.

(C) Ref. [17] has also brought up 1970 lecture notes by Zumino [19], where Wess and Zumino had proposed four models of a massive spin-2 hadron,  $f_{\mu\nu}$ , interacting with Einstein's gravity for  $g_{\mu\nu}$ . Using the recently developed methods in [4, 5, 20, 21, 22, 23], we can tell now that the two (interrelated) “f-g” models of Wess and Zumino are ghost-free as full non-linear theories and represent a subset of more general ghost-free bi-gravities [21]; if reduced to a single massive graviton, they represent a subset of the ghost-free massive gravities [5], from which [21] originates. The two other theories proposed in [19] have Boulware-Deser (BD) ghosts [24]. None of the conclusions on the absence or presence of the BD ghost can be deduced from Ref. [19] itself, not surprisingly, as it precedes the BD work [24]. More details will be given elsewhere.

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<sup>1</sup>One example, where superluminality was claimed to be removed, at least for spherically symmetric sources at the classical level, is a bi-galileon theory [25]. This requires a low energy remnant – the second Galileon which is massless and operates at arbitrarily low scales. We note that certain bi-galileons naturally emerge in generalization of ghost-free massive gravity [5] to a theory with a dilaton-like particle, quasidilaton [26]. Although the issue of superluminality for the quasidilaton has not been studied yet, predictions of this theory are not brought in conflict with observations by the presence of the low-energy remnant, the quasidilaton [26].

## 2 The Theory

In the decoupling limit of massive gravity, the Lagrangian describing helicity-2 states, contained in  $h_{\mu\nu}$ , and helicity-0 state, denoted by  $\pi$ , with their coupling to matter stress-tensor  $T_{\mu\nu}$ , is given by [4]:

$$\mathcal{L} = -\frac{1}{2}h^{\mu\nu}\mathcal{E}_{\mu\nu}^{\alpha\beta}h_{\alpha\beta} + h^{\mu\nu}\left(X_{\mu\nu}^{(1)} + \frac{\alpha}{\Lambda_3^3}X_{\mu\nu}^{(2)} + \frac{\beta}{\Lambda_3^6}X_{\mu\nu}^{(3)}\right) + \frac{1}{M_{\text{pl}}}h^{\mu\nu}T_{\mu\nu}, \quad (1)$$

where, the three conserved symmetric tensors  $X_{\mu\nu}^{(n)}(\Pi)$  depend on the second derivatives of helicity-0 field  $\Pi_{\mu\nu} \equiv \partial_\mu\partial_\nu\pi$ , and can be cast as follows [27],

$$\begin{aligned} X_{\mu\nu}^{(1)} &= -\frac{1}{2}\varepsilon_\mu^{\alpha\rho\sigma}\varepsilon_\nu^{\beta}{}_{\rho\sigma}\Pi_{\alpha\beta}, \\ X_{\mu\nu}^{(2)} &= \frac{1}{2}\varepsilon_\mu^{\alpha\rho\gamma}\varepsilon_\nu^{\beta\sigma}{}_\gamma\Pi_{\alpha\beta}\Pi_{\rho\sigma}, \\ X_{\mu\nu}^{(3)} &= \frac{1}{2}\varepsilon_\mu^{\alpha\rho\gamma}\varepsilon_\nu^{\beta\sigma\tau}\Pi_{\alpha\beta}\Pi_{\rho\sigma}\Pi_{\gamma\tau}, \end{aligned}$$

where  $\varepsilon_{\mu\nu\alpha\beta}$  is the usual Levi-Civita symbol.

Under the linear diffeomorphisms  $\delta h_{\mu\nu} = \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu$ , the full nonlinear Lagrangian (1) is invariant up to a total derivative, while it is exactly invariant under the field space galilean transformations,  $\delta\pi = v_\mu x^\mu$ , with constant  $v_\mu$ .

Due to the specific structure of this Lagrangian, the coefficients  $\alpha$  and  $\beta$  do not get renormalized by quantum loops; hence, any choice of their values is technically natural [29].

For generic values of these coefficients the helicity-2 and helicity-0 modes mix, as seen from (1); this mixing cannot be undone at the full nonlinear level, unless  $\beta = 0$  [4]. The choice  $\beta = 0$  corresponds to selecting a special relation,  $\alpha_3 = -\alpha_4/4$ , between the coefficients  $\alpha_3$  and  $\alpha_4$  of the cubic and quartic  $K$  terms of the full massive theory [5].

For  $\beta = 0$ , which as already mentioned is a technically natural choice stable under loop corrections, there exists an invertible field redefinition

$$h_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} + \pi\eta_{\mu\nu} + \frac{\alpha}{\Lambda_3^3}\partial_\mu\pi\partial_\nu\pi, \quad (2)$$

that decouples the tensor and scalar modes from each-other [4]. The Lagrangian (1) then takes the form  $\mathcal{L} = \mathcal{L}_{\bar{h}} + \mathcal{L}_\pi$ , where  $\mathcal{L}_{\bar{h}}$  is the linearized Einstein-Hilbert action for  $\bar{h}$  minimally coupled to  $T_{\mu\nu}$ , while

$$\begin{aligned} \mathcal{L}_\pi &= \frac{3}{2}\pi\Box\pi + \frac{3}{2}\frac{\alpha}{\Lambda_3^3}(\partial\pi)^2\Box\pi + \frac{1}{2}\frac{\alpha^2}{\Lambda_3^6}(\partial\pi)^2([\Pi^2] - [\Pi]^2) \\ &\quad + \frac{1}{M_{\text{pl}}}\pi T + \frac{\alpha}{M_{\text{pl}}\Lambda_3^3}\partial_\mu\pi\partial_\nu\pi T^{\mu\nu}. \end{aligned} \quad (3)$$

Here, square brackets  $[\cdot]$  denote the trace. Note that the absolute value of the only free parameter  $\alpha$  is immaterial, as it can be absorbed into the scale  $\Lambda_3$ , however, the sign of  $\alpha$  will be crucial [rescaling  $\Lambda_3$  would correspond to rescaling of the graviton mass; for comparisons with massive gravity, however, we will manifestly keep the parameter  $\alpha$ , and unrescaled  $\Lambda_3$  in what follows].

Thus, for  $\beta = 0$ , the dynamics of the helicity-0 mode is similar to the cubic and quartic Galileons, but with some special coefficients in front of these terms; moreover, the last term in the second line in (3) is a coupling to stress-tensor that is not usually considered in the Galileon theories, which however is present in massive gravity [6, 4, 7], (it will play a crucial role). As mentioned in the previous section, we refer to (3) as Restricted Galileons, and discuss this theory in the present work. The case with  $\beta \neq 0$ , which appears to be quantitatively different, will be discussed in Ref. [30].

As to the parameter  $\alpha$ , *a priori* it can have the either sign. However, the last term in (3), leads to classical renormalization of the kinetic term for  $\pi$ . For a negative sign of  $\alpha$  this renormalization is negative, and for most of the reasonable sources it would overshoot the sign of the  $\pi$  kinetic term. Furthermore, we will show in Appendix A, that classical renormalization of the  $\pi$  kinetic term due to the non-linear terms in (3), although positive, is in fact subdominant. Therefore  $\alpha < 0$  case is not physical: when a localized source (with density greater than  $\Lambda_3^3 M_{\text{pl}}$ ) is approached from far away, at some point the  $\pi$  field will have vanishing kinetic term, signaling an infinitely strongly coupled regime; at yet shorter distances the kinetic term would flip its sign (if one could pass into this region), to convert  $\pi$  into a ghost. For this reason, our physical choice will be  $\alpha > 0$ .

## 2.1 Spherically symmetric solutions

In this work we study a gravitational field configuration created by a source of finite size  $R$ , and a uniform density  $\rho$ . The pressure of the source will also have some important consequences, and will be discussed in Appendix B.

Without the loss of generality, the static spherically symmetric configuration can be parametrized by the following ansatz for the metric perturbations

$$h_{00} = a(r), \quad h_{ij} = f(r)\delta_{ij}, \quad (4)$$

while for the helicity-0 we begin by assuming the radial ansatz  $\pi = \pi(r)$ . Then, the equations of motion for graviton reduces to two ordinary differential equations (this is after integrating them once and requiring the solution to vanish at the origin)

$$r f' = -\frac{2M}{M_{\text{pl}} r} + \Lambda_3^3 r^2 \lambda (1 - \alpha \lambda), \quad r a' = -\frac{2M}{M_{\text{pl}} r} - \Lambda_3^3 r^2 \lambda, \quad (5)$$

where above, and in what follows we use the notations for  $\lambda$  and the Vainshtein

radius  $r_*$ :

$$\lambda \equiv \frac{\pi'}{\Lambda_3^3 r}, \quad r_* \equiv \left( \frac{M}{M_{\text{pl}}^2 m^2} \right)^{1/3}, \quad (6)$$

and the prime denotes differentiation w.r.t. the radial coordinate  $r$ .

The closed form of the equations of motion for the helicity-0 mode can be obtained by integrating out  $a$  and  $f$ , using (5). The net result reads as follows [28]

$$3\lambda - 6\alpha\lambda^2 + 2\alpha^2\lambda^3 = \begin{cases} 2\left(\frac{r_*}{r}\right)^3 & r > R \\ 2\left(\frac{r_*}{R}\right)^3 & r < R \end{cases}. \quad (7)$$

We could have also derived this latter equation by simply integrating the Galileon equations of (3), with the same initial conditions.

It is now straightforward to show that for  $\alpha > 0$  equation (7) has no solution that could interpolate from the Vainshtein region to an asymptotically flat one, i.e., to the region where  $\lambda \rightarrow 0$ . To see this we look at the cubic polynomial in  $\lambda$  at the lhs of (7). The necessary and sufficient condition for the solution of (7) to interpolate between the Vainshtein region and the asymptotically flat one, is that for the polynomial on the lhs of (7) to have only one zero, at  $\lambda = 0$  [3]. However, this condition is not satisfied if  $\alpha > 0$ : there are three real zeros in this case,  $\lambda_1 = 0, \lambda_{2,3} = (3 \pm \sqrt{3})/2\alpha$ . This means that the solution that starts off in the Vainshtein region, where  $\lambda \gg 1$ , matches onto a solution that at  $r \rightarrow \infty$  tends to  $\lambda_2 = \frac{3+\sqrt{3}}{2\alpha}$ . The latter implies that asymptotically  $\pi \rightarrow \Lambda_3^3 r^2 \lambda_2$ .

Then, the solution to (5)-(7) for  $\alpha > 0$ , in the leading approximation, is:

*Outside the Vainshtein radius*

$$\lambda \simeq \frac{3 + \sqrt{3}}{2\alpha}, \quad a \simeq -\frac{\Lambda_3^3 r^2}{2} \lambda, \quad f \simeq \frac{\Lambda_3^3 r^2}{2} \lambda (1 - \alpha \lambda). \quad (8)$$

*Inside the Vainshtein radius*

$$\lambda \simeq \frac{r_*}{\alpha^{2/3} r} + \frac{1}{\alpha} + \frac{r}{2\alpha^{4/3} r_*}, \quad a \simeq f \simeq -\frac{2M}{M_{\text{pl}} r} + \mathcal{O}(r^2 \lambda). \quad (9)$$

This solution describes the static spherically symmetric source of mass  $M$  on a cosmological background, with the equation of state  $p/\rho = -2/3 + 1/\sqrt{3} \approx -0.1$ .

Next, we would like to proceed with the analysis of fluctuations around this solution. As it is clear from the above, we gave up the asymptotic flatness. Therefore, it is natural to begin by studying the region far away from the source. Due to the decoupling between  $\bar{h}$  and  $\pi$  we can analyze their fluctuations separately. Those of  $\bar{h}$  are linear GR perturbations on a cosmological background with the above-mentioned equation of state, taken in the high momentum approximation (momenta being larger than the Hubble parameter, which in this case is  $\sim m$ ). These fluctuations are stable, and there is no novelty in this part.

Let us then look at the fluctuations of  $\pi$ , which we decompose as  $\pi = \pi_{cl}(r) + \sigma(x, t)$ , where  $\pi_{cl}(r)$  denotes the classical background just found in (8). Appendix A gives an expression for the Lagrangian of fluctuations about a general background  $\Phi$ ; substituting  $\Phi = \pi_{cl}$  into (A-II), we obtain the quadratic Lagrangian for fluctuations around (8):

$$\mathcal{L} = 3 \left[ 5 + 3\sqrt{3} \right] (\partial_t \sigma)^2 - \frac{3}{2} \left[ 1 + \sqrt{3} \right] (\partial_j \sigma)^2. \quad (10)$$

Hence, the cosmological background is stable against all linear perturbations. Moreover, the fluctuations propagate with sub-luminal velocities.

As a next step we study fluctuations in the presence of the source of mass  $M$ . Outside the Vainshtein radius the just-studied cosmological background dominates, and hence the fluctuations are stable and sub-luminal. However, inside the Vainshtein radius the source defines the background and its effects are dominant. The fluctuations of  $\bar{h}$  coincide, to a good approximation, with those of linearized GR on a background of a static spherically symmetric source. The fluctuations of  $\pi$ , however, need an extra caution. Using again (A-II), we find that in the leading order in  $r/r_*$  (which is a good approximation in the Vainshtein regime where  $(r/r_*) \ll 1$ ), the small excitations around (9) are described by the following Lagrangian:

$$\mathcal{L} = 3 \left[ \alpha^{2/3} \left( \frac{r_*}{r} \right)^2 + 2\alpha^{1/3} \frac{r_*}{r} \right] (\partial_t \sigma)^2 - 3\alpha^{2/3} \left( \frac{r_*}{r} \right)^2 (\partial_r \sigma)^2 - \frac{3}{2} (\partial_\Omega \sigma)^2. \quad (11)$$

Remarkably, radial fluctuations propagate with a slightly sub-luminal speed –the deviation from the luminality being suppressed by  $r/r_*$ . The angular speed of sound, on the other hand, is strongly suppressed. This suppression, for a generic quartic Galileons, was already emphasized in Ref. [3].

It is interesting to ask what happens with a dilute distribution of matter with some size  $R$ , and density below  $\Lambda_3^3 M_{\text{pl}}$ , placed on an asymptotically flat space. Such a source has no Vainshtein region, since  $r_* < R$ , and there is a static solution with decaying  $\pi$  field at infinity. However, if the source is adiabatically collapsed to form a smaller object with density above  $\Lambda_3^3 M_{\text{pl}}$  (i.e., with  $r_* > R$ ), then the energy density and pressure in the nonlinear  $\pi$  field blows up when  $r_* \sim R$ , preventing such an adiabatic collapse.

### 3 Observational prospects ?

In this section we will consider a hypothetical universe described by the massive gravity with  $\beta = 0$ . This is a technically natural choice, as it's not ruined, to a high degree of accuracy, by loop corrections [29]. We take the solution (8) as our background in an empty space, anticipating that localized sources will match onto this solution, and not to a solution with flat asymptotics.

### 3.1 Nonperturbative effects in Solar System

Since the background (8) has an unusual equation of state, we will require this fluid to be a subdominant component in the universe<sup>2</sup>. This can be arranged by choosing the parameter  $\alpha$  to be of order ten or so (for a fixed graviton mass  $m \sim H_0$ ), which again, is a technically natural choice [29]. Then, to obtain realistic universe we would have to introduce dark energy, and matter density, roughly as we do in the  $\Lambda$ CDM model. Therefore, from the point of view of getting dark energy from modified gravity, the  $\beta = 0$  theory is not interesting, however, our goal here is to understand if such a theory, in principle, could be consistent with observational data.

To this end we would like to discuss observational constraints due to the small corrections to the GR results within the Vainshtein radius [8, 31, 32]. From the solution (9) we easily deduce that the fractional increase of the Newtonian potential inside the Vainshtein radius is

$$q \equiv \frac{\Delta\phi}{\phi} \simeq \frac{m^2 r_* r / \alpha^{2/3}}{r_g / r}, \quad (12)$$

where by  $\phi$  we denote the Newtonian potential. This deviation from the standard potential will give rise to an additional perihelion advancement of orbiting bodies (an additional to the GR effects). For careful account of these effects, within a relativistic theory, see [31, 34].

Then, the question is whether such deviations can be measured. One of the most precisely measured trajectory is that of the Moon orbiting the Earth [35]. For the Earth-Moon system, and for  $m \simeq H_0$ , the fraction (12) is of order  $q^{Earth-Moon} = 10^{-16} / \alpha^{2/3}$ ; this is four orders of magnitude below the same fraction in the DGP model [31, 32] (see, also [33]); as such its influence on the perihelion advancement of the Lunar orbit is completely negligible and is unlikely to be measurable by the near future Lunar Laser Ranging Experiments [35].

Next, we discuss whether fluctuations of the  $\pi$  field, above its background, can give rise to additional forces competing with gravity. There is an important point, in these considerations, as compared to conventional Galileons. The kinetic term for the fluctuations  $\sigma = \pi - \pi_{cl}$  gets additional classical renormalization

$$- \left( \eta_{\mu\nu} + Z_{\mu\nu}^V + \frac{\alpha T_{\mu\nu}}{M_{\text{pl}} \Lambda_3^3} \right) \partial^\mu \sigma \partial^\nu \sigma + \frac{1}{M_{\text{pl}}} \sigma T, \quad (13)$$

where the factor  $Z^V$  is due to the cubic and quartic Galileon terms evaluated on the solution; we have also manifestly shown couplings of the fluctuations to the stress-tensor. Note that the kinetic term in (13) gets enhanced by both a factor due to the classical background,  $Z^V$ , and by a source-dependent term  $\alpha T$  (both of these are huge factors for realistic sources, but as shown in Appendix A, the

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<sup>2</sup>Although, other more interesting options may also be possible when the density of this fluid adds in a meaningful way to the matter density, while its small negative pressure, adds a bit to the negative pressure due to dark energy.



enhancement due to the  $\alpha T$  term dominates within a source). As a result, the static force due to exchange of  $\sigma$ , between any two objects, say two small metallic balls, is suppressed by a tremendous factor determined by the ratio of density and pressure in the balls over the density  $\Lambda_3^3 M_{\text{pl}}$ ; the latter happens to coincide with the critical density,  $\rho_c \simeq 10^{-29} g/cm^3$ , if  $m \sim H_0 \simeq 10^{-33} eV$ . This fraction is enormous,  $(\rho_{\text{metall}}/\Lambda_3^3 M_{\text{pl}}) \gtrsim 10^{30}$ . As we will discuss in the next section, this suppression makes the  $\sigma T$  coupling unobservable. Therefore, the exchange due to a single  $\sigma$  field cannot be gravity-competing force. In the next section we turn to quantum effects, and the question whether they could ruin the above conclusion.

### 3.2 Comments on quantum effects

In this section we would like to discuss quantum loop effect. Before we do so, one should point out that we're dealing with a theory that is non-renormalizable by power-counting, and at least in this sense, requires a completion at the scale  $\Lambda_3$ . However, judging from the properties of the theory below  $\Lambda_3$  this completion, unlike in massive non-Abelian vector fields, is not expected to be a conventional one (see [36, 29], and references therein). On the other hand, discussions in the literature, as well as below, are based on an assumption of a certain conventional generic completion. This is an assumption that may not be a right one, and therefore, all such discussions are on a shaky ground.

Putting the above concern aside, we would like to analyze the viability of the bound imposed on graviton mass in Ref. [11], based on the conventional effective field theory considerations of the decoupling limit Lagrangian [37]. To do so we first briefly recall what do the quantum corrections lead to in the decoupling limit of massive gravity [29]: the unambiguous result, independent of the uncertainties of the previous paragraph, is that the terms presented in (1) do not get renormalized! However, in a conventional quantum effective field theory approach new terms may get induced in the 1PI action. These new terms include ambiguous power-divergent terms, as well as log divergent pieces that carry information about the forward scattering of the quanta, and have to be included in the 1PI action. What was argued for the cubic [37], as well as generic Galileons [3], is that these terms, in the conventional approach, and once calculated on a classical background, end up being suppressed by a scale of the classical background itself.

The classical renormalization (13) increases the effective energy/momentum scale of strong interactions of the fluctuations in the 1PI action [37]. The naive analysis shows that the loop-induced terms are suppressed by the effective scale schematically written as follows:

$$\Lambda_{\text{eff}} \simeq (Z^V + \frac{\alpha T}{M_{\text{pl}} \Lambda_3^3})^{1/3} \Lambda_3 \equiv Z_{\text{tot}}^{1/3} \Lambda_3. \quad (14)$$

This is valid when the enhancements of the kinetic and gradient terms are of the same order; however, (14) fails to describe the strong coupling scale in case of the

hierarchy among them. For the illustration, consider the following toy model, that captures essential features of our theory on a static background:

$$\mathcal{L} = a^2(\partial_t\pi)^2 - b^2(\partial_j\pi)^2 + \frac{b_*}{\Lambda_3^3}(\partial\pi)^2\Box\pi, \quad (15)$$

where  $a, b$  and  $b_*$  are assumed to be constants for simplicity. The kinetic term for the field  $\pi$  needs to be canonically re-normalized,  $\pi \rightarrow \pi/a$ . This results in the naive  $\Lambda_{eff} \sim ab_*^{-1/3}\Lambda_3$  estimate for the effective cut-off, in the spirit of (14). However, by calculating the  $2 \rightarrow 2$  scattering amplitude one can see that the strong coupling scale differs in different channels; this is because the propagator has the denominator,  $\omega^2 - (b/a)^2\vec{k}^2$ . In fact, for the forward scattering and  $b \ll a$ , the  $u$ -channel is the strongest. It gives rise to the  $\Lambda_{eff} \sim (b/b_*)^{1/3}a^{2/3}\Lambda_3$  cut-off, while the  $s$ -channel amplitude becomes strong only at  $ab_*^{-1/3}\Lambda_3$  scale; the latter being considerably larger than the former.

Higher order tree-level diagrams give higher effective scale: the vertices have more powers of inverse  $a$  than the propagators can compensate for due to the  $(b/a)^2$  factor in the diagrams that aren't automatically zero<sup>3</sup>. All this remains to be the case even after the quartic Galileon is included in (15). Hence, the effective lowest strong coupling scale remains to be:

$$\Lambda_{eff} = a^{2/3} \left( \frac{b}{b_*} \right)^{1/3} \Lambda_3. \quad (16)$$

At this scale, interactions of the fluctuations would become non-perturbative. Let us now discuss how this squares with the fact that gravity-competing forces have been excluded down to the scales of order 100 microns or so [40]. First, recall that in the DGP model,  $\Lambda_{eff}^{-1}$  is of order a centimeter [37], the scale much bigger than 100 microns. However, as pointed out in [37] this is not likely to be an issue, since these fluctuations are very weakly coupled to matter sources, and thus, their effects are unlikely to be visible in the submillimeter measurements, even though they themselves self-interact strongly at scales below 1cm.

For the model considered in the present work, it was argued in Ref. [11] that the  $\Lambda_{eff}^{-1}$ , is much bigger than a millimeter, if the graviton Compton wavelength is taken to be of the Hubble size. Does this mean that such a small graviton mass is ruled out by laboratory measurements, as claimed in [11]?

To address this question in detail, consider a device measuring putative gravity-competing forces – a torque pendulum of the Adelberger's group experiment [40]. Then there are two important points:

(1) Within the matter that the pendulum plates are made of the effective strong coupling scale differs significantly from its value outside of the plates. Using (16),

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<sup>3</sup>The power divergent terms in the loops, that could induce lower cutoff, are removed by the counterterms as per the general philosophy adopted in this approach; the log divergent terms however, are related to the tree-level diagrams by unitarity and analyticity. Hence, it suffices to get the lowest cutoff of the tree-level diagrams as we did above.

and the fact that in this case  $a \sim 10^{15}$ ,  $b \sim b_*$ , we find that in the plates  $\Lambda_{eff}^{-1} \simeq 10^{-10} \text{ 1000 km} \simeq 0.1 \text{ mm}$ . This is smaller than the thickness of the upper and lower pendulum plates, that respectively are about 1.8 and 7.8 mm [40]<sup>4</sup>.

(2) Thus, in the torque pendulum plates, the fluctuations are weakly coupled to themselves, and they can be well-described by the classical Lagrangian down to 0.1 millimeters. On the other hand, as shown in the previous section, this classical Lagrangian gives rise to a tremendous suppression of the coupling  $\sigma T/M_{\text{pl}}$ , that determines how strong/weak the coupling of the fluctuations to the plates could be. For the metallic plates this suppresses the  $\sigma T/M_{\text{pl}}$  vertex by an additional factor of  $a^{-1} \sim 10^{-15}$ !

Therefore, a single  $\sigma$  fluctuation cannot efficiently be emitted by the plates (or any realistic source bigger than 0.1 millimeter, for that matter). Even though the fluctuations in the vacuum between the plates become strongly interacting with themselves, and could form some bound states of  $\sigma$ , what's important is that this exotic strongly self-coupled sector cannot couple efficiently to the measuring device – the latter coupling due to the exchange of  $\sigma$  is  $a^{-2}(b/a)^{-2} \sim 10^{-18}$  of the strength of gravity. Therefore, the exchange due to  $\sigma$  cannot be gravity-competing force – in spite of the fact that the  $\sigma$  field self-interacts strongly outside of the plates. This suppression was not taken into account in Ref. [11], making the bound imposed on the graviton mass by that work unwarranted. The valid phenomenological bounds on the graviton mass are given in Ref. [42].

Note that for  $\beta \neq 0$ , which was also considered in [11], the effect of the suppression of coupling to matter, due to the  $\partial^\mu \sigma \partial^\nu \sigma T_{\mu\nu}$  term, remains valid. However, there are technical and conceptual differences for  $\beta \neq 0$  [30], that obfuscate the results of Ref. [11] for that case; this and related issues will be presented in detail in [30].

In conclusion of this section, and to reiterate, we note that many statements about the quantum effective theory are based on assumptions about what the UV completion might be. It is clear from various standpoints (see, e.g., [29] and references therein), that theories of massive gravity need a completion, or alternatively, a dual formulation (perhaps along the lines of [38, 39]), at and above the  $\Lambda_3$  scale. However, judging from the properties of these theories (such as e.g., nonrenormalizability of some of the couplings [29]) the putative completion should not be expected to be of a conventional and generic type, as preassumed in all the above discussions. Therefore, conclusions deduced from such considerations [11], even if they were plausible, would not be set in stones.

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<sup>4</sup>This scale is greater than the width of the plates, 0.3 microns, used in the experiment of Ref. [41] that exclude forces 14,000 times stronger than gravity at 10 microns. However, the coupling to matter that we're discussing is  $10^{-18}$  of the coupling of gravity at 0.1mm (see below), and it would be unreasonable to expect for it to become 14,000 times stronger than gravity at the micron scale.

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## Appendix A

In order to study the stability, we split the scalar degree of freedom into the background  $\Phi$  and the fluctuation  $\sigma$  as follows

$$\pi = \Phi + \sigma. \quad (\text{A-I})$$

As a result, to the second order in perturbations, the Lagrangian (3) becomes<sup>5</sup>

$$\begin{aligned} \mathcal{L}_\sigma = & \left\{ \left[ -\frac{3}{2} + 3\frac{\alpha}{\Lambda_3^3} \square \Phi + \frac{3}{2} \frac{\alpha^2}{\Lambda_3^6} ((\partial_\alpha \partial_\beta \Phi)^2 - (\square \Phi)^2) \right] \eta_{\mu\nu} \right. \\ & + \left[ -3\frac{\alpha}{\Lambda_3^3} \partial_\mu \partial_\nu \Phi + 3\frac{\alpha^2}{\Lambda_3^6} (-\partial_\alpha \partial_\mu \Phi \partial_\alpha \partial_\nu \Phi + \square \Phi \partial_\mu \partial_\nu \Phi) \right] \Big\} \\ & \times \partial_\mu \sigma \partial_\nu \sigma + \frac{1}{M_{\text{pl}}} \sigma T + \frac{\alpha}{M_{\text{pl}} \Lambda_3^3} \partial_\mu \sigma \partial_\nu \sigma T^{\mu\nu}. \end{aligned} \quad (\text{A-II})$$

For the static clump of dust of constant density, that is for  $T_{\mu\nu} = \rho \delta_\mu^0 \delta_\nu^0 \theta(R-r)$ , we obtain the following kinetic term (to the leading order) inside the source

$$\left[ \alpha \frac{\rho}{M_{\text{pl}} \Lambda_3^3} + 9 \left( \frac{\alpha^{1/3} r_*}{R} \right)^2 \right] (\partial_t \sigma)^2. \quad (\text{A-III})$$

Here, the second term in brackets comes from the quartic Galileon term after we have substituted the classical solution within Vainshtein region (9). After the replacement  $\rho \rightarrow M/R^3$ , (A-III) leads to the conclusion that in  $\alpha < 0$  parameter space the scalar perturbations have ghost-like kinetic term within the source of radius  $R \ll \alpha^{1/3} r_*$ . The latter condition can be translated on the language of the source density as

$$\alpha \rho > M_{\text{pl}} \Lambda_3^3. \quad (\text{A-IV})$$

For phenomenologically interesting value of  $\Lambda_3^{-1} \sim 1000 \text{ km}$  the above bound becomes  $\alpha \rho > 10^{-29} \text{ g/cm}^3$ .

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<sup>5</sup>Here we give only the scalar part of the Lagrangian, since the tensor mode is completely decoupled and propagates according to the linearized Einstein-Hilbert action.

As we've shown, for  $\alpha > 0$ , the asymptotically flat solution cannot be continued into the Vainshtein region. However, there are cosmological solutions that are free of the above difficulty.

In Section 2 we have already presented the static solution which asymptotes to the cosmological background, rather than the Minkowski space. Here we consider the time-dependent ansatz:

$$\pi = \frac{c}{2}\Lambda_3^3 t^2 + \pi_0(r), \quad (\text{A-V})$$

The introduction of non-zero  $c$  modifies (7) in the following way

$$\begin{aligned} & 3(1 + 2\alpha c)\lambda - 6(\alpha + \alpha^2 c)\lambda^2 + 2\alpha^2 \lambda^3 \\ &= \begin{cases} 2\left(\frac{r_*}{r}\right)^3(1 + 2\alpha c) + c, & r > R \\ 2\left(\frac{r_*}{R}\right)^3(1 + 2\alpha c) + c, & r < R \end{cases} \end{aligned} \quad (\text{A-VI})$$

The above-mentioned static cosmological background corresponds to  $c = 0$  in (A-V). However, there is a plethora of available backgrounds, which correspond to different values of  $c$  and the only discriminating principle among various backgrounds must be their stability. In this letter, we do not pursue the detailed analysis of all the possible solutions. Merely, we would like to present one more solution which in our opinion is quite special.

From (A-VI), it follows that there is a peculiar value of  $c = -1/(2\alpha)$ , which makes the equation independent of the mass of the source. As a result, eq.(A-VI) has three solutions. One of them being  $\lambda = 1/(2\alpha)$ , which corresponds to an unstable self-induced de Sitter space found in [27] (it propagates a ghost) and we discard it. The other two are quite interesting

$$\lambda = \frac{1 \pm \sqrt{3}}{2\alpha}, \quad (\text{A-VII})$$

in particular the pion cloud has zero pressure and positive energy. The easiest way to see this, is to look at the metric itself

$$a = f = \frac{2M}{M_{\text{pl}}r} - \frac{\Lambda_3^3}{4\alpha}r^2. \quad (\text{A-VIII})$$

The effective pressure takes the following form

$$p = \frac{M_{\text{pl}}}{3}G_i^i = -\frac{M_{\text{pl}}}{6}\varepsilon_i^{\alpha\rho\gamma}\varepsilon^{i\beta\sigma}{}_\gamma\partial_\alpha\partial_\beta h_{\rho\sigma} = \frac{M_{\text{pl}}}{3}\Delta(f - a) = 0. \quad (\text{A-IX})$$

Moreover, the energy density of the pion fluid, filling the space, is given by

$$\rho = M_{\text{pl}}G_{00} = -M_{\text{pl}}\Delta f = \frac{3M_{\text{pl}}\Lambda_3^3}{2\alpha} > 0, \quad \text{for } \alpha > 0. \quad (\text{A-X})$$

On this background, the quadratic Lagrangian for perturbations is

$$\mathcal{L} = 6(\partial_t \sigma)^2 - \frac{3}{2}(\partial_j \sigma)^2, \quad (\text{A-XI})$$

everywhere in the space. As it is easy to see, the fluctuations are stable and propagate with the half speed of light.

We would like to emphasize that the  $\pi$  profile is independent of the source

$$\pi = \frac{\Lambda_3^3}{4\alpha} \left[ -t^2 + \left( 1 + \sqrt{3} \right) r^2 \right]. \quad (\text{A-XII})$$

This means, that the presence of the static, spherically symmetric and pressure-less source does not excite the longitudinal mode of the massive graviton.

## Appendix B

Here, we would like to discuss the effect of pressure ( $p \ll \rho$ ) on the background and its stability. The reason is simple, most of the sources in the universe possess small, yet nonzero pressure.

For simplicity, we treat only  $c = 0$  case here, since the generalization is straightforward. The equation of motion for the longitudinal mode (7) takes the following form

$$\begin{aligned} & 3\lambda - 6\alpha\lambda^2 + 2\alpha^2\lambda^3 \\ &= \begin{cases} 2\left(\frac{r_0}{r}\right)^3 & \text{Outside the source} \\ 2\left(\frac{r_*}{R}\right)^3 \left[ 1 + \frac{p}{\rho}(\alpha\lambda - 3) \right] & \text{Inside the source} \end{cases}, \end{aligned} \quad (\text{B-I})$$

where  $r_0$  is an integration constant, which is determined by the matching condition at the surface of the source. As expected, the only change in the field configuration is the effective mass.

Let us recall that we have chosen the sign of  $\alpha$  to be positive to avoid ghosts inside the source. The danger was coming from the last term of (A-II). But, if we avoid the ghost by appropriate choice of  $\alpha$  then for nonzero pressure that term leads to the tachyonic contribution to the gradient energy. However, after taking a closer look, it can be shown that the healthy contribution to the gradient energy, coming from the quartic Galileon, overwhelms the unhealthy one; rendering the background stable. A simple estimate shows that this remains to be the case as long as,  $\rho \gg p$ . For  $p \sim \rho$ , on the other hand, the coefficient of the gradient term of the perturbations is proportional to  $(\rho - 2p)/p$ , and for  $p > \rho/2$  flips the sign to yield gradient instabilities. Such instabilities should manifest themselves in production of the  $\pi$  quanta (albeit the coupling of these quanta to the matter is very weak). Whether this may be dangerous, or interesting, for high density astrophysical objects, such as neutron stars, should be investigated including the fact that they're rotating and are neither exactly spherically symmetric, nor homogeneous; that is, they lack properties used in the derivation of above-mentioned results.

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